

# A solution to Higgs naturalness

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## Abstract

The Standard Model (SM) is usually considered to be unnatural because the scalar Higgs mass receives a quadratic divergent correction. We suggest a new way to solve the naturalness problem from point of view of renormalization group method. Our approach is illustrated through the familiar  $\phi^4$  theory. A renormalization group equation for scalar field mass is proposed by introducing a subtraction scale. We give a non-trivial prediction: the Higgs mass at short-distance is a damping exponential function of the energy scale. It follows from a characteristic of the SM that the couplings to Higgs are proportional to field masses, in particular the Higgs self-interactions. In the ultraviolet limit, the Higgs mass approaches to a mass called by Veltman mass which is at the order of the electroweak scale. The fine-tuning is not necessary. The Higgs naturalness problem is solved by radiative corrections themselves.

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## I. INTRODUCTION

The Higgs field plays a fundamental role in the SM. It provides the origin of spontaneous symmetry breaking and masses of all matter fields. The crucial purpose of the running Large Hadron Collider (LHC) is to test this mechanism. However, it was known for a long time that the scalar field suffers from a problem caused by quadratic divergence [1, 2]. In particular, the one-loop correction to the Higgs mass square, is proportional to a large momentum cut-off  $\Lambda^2$  by [3, 4]

$$m_H^2 = (m_H^0)^2 + \frac{3}{8\pi^2 v^2} [m_H^2 + 2m_W^2 + m_Z^2 - 4m_t^2] \Lambda^2 . \quad (1)$$

where  $m_i$  are masses of gauge and fermion fields. In order to satisfies the experimental constraints on the Higgs mass which is at the order of 100 GeV, a delicate cancelation between the bare mass square and the counter-term requires an incredible fine-tuning of parameters. Because there is no symmetry protecting the small Higgs mass in the SM, the Higgs is considered as "unnatural".

Many proposals have been proposed to solve the naturalness problem [5]. Veltman pointed out a relation [3]

$$m_H^2 = 4m_t^2 - 2m_W^2 - m_Z^2 . \quad (2)$$

If the above condition holds, the quadratic divergence cancels. Taking into account of higher order corrections, the Veltman condition is no longer valid [4]. Supersymmetry has an attractive property to solve the naturalness problem by cancelation between fermions and bosons [6]. But this symmetry is broken in reality, and the Higgs mass depends on the supersymmetry breaking scale quadratically. In [7], the authors suggest that the cancelation of quadratic divergence at scale of new physics.

The result given in Eq. (1) contains only the one-loop contribution. Although seemingly quadratically divergent, it is not by all means that the final result with all-order radiative corrections are divergent. In quantum field theory, there are some unexpected or non-trivial examples which contradicts the simple intuition. One classical example is the asymptotic freedom. The coupling constant of a non-Abelian gauge field theory, e.g. QCD, is logarithmic divergent in one-loop, while it vanishes in the short-distance limit. Another example is the elastic form factor of a fermion at large momentum transfer which is usually called by Sudakov form factor [8, 9]. The one-loop correction contains a large double-logarithm. Summing double-logarithm to all orders in the coupling constant produces a rapid damping exponential function. The success of the two examples relies on renormalization group method. Since the SM is renormalizable, the Higgs mass is independent of the cut-off scale, and Eq. (1) does not provide the true scale dependence. Moreover, whether the bare mass is really divergent or not is unknown. To answer these questions, it is necessary to study the renormalization evolution and the short-distance behavior of the Higgs mass.

The dimensional regularization is the most popular method to regulate the divergence, while the quadratic divergence is absent in this method because of a definition that scaleless integral is zero. According to this point, some theorists have the opinion that there is no naturalness problem at all. In [3], Veltman pointed out that dimensional regularization is not physical because theory with space-time dimension  $d \neq 4$  is unphysical. We provide another comment based on Wilson's renormalization group method [10]. The dimensionless integral  $\int \frac{d^4 k}{k^2}$ , which is quadratic divergent by dimensional analysis, sums up the virtual particle contributions with momentum square from 0 to  $\infty$ . The larger the momentum square is, the more important it contributes. If we simply defined this integral to be zero, the real physics from different energy scales will be missed. The conventional renormalization group equations are usually given in dimensional regularization where the mass renormalization is multiplicative. Because of the quadratic divergence, the renormalization of mass is additive. We have to search for new types of renormalization group equation.

Nevertheless, a consistent renormalization program for the quadratic divergence, such as the regularization method and renormalization scheme, is not mature. Veltman uses the dimensional regularization, but chooses the dimension  $d$  close to 2 rather than 4 in the integral because the pole occurs at  $d = 2$ . Then he defines the pole to be proportional to the momentum cut-off square [3]. This treatment seems to be a combination of dimensional regularization and momentum cut-off. Another obstacle concerns the renormalization scheme, i.e. the choice of renormalization condition. Fujikawa proposed a speculative scheme by introducing a subtraction scale [11]. This scheme is simple and has an advantage in deriving the renormalization group equations. We will discuss this method and use it to discuss the evolution of scalar field mass. The implications to the Higgs naturalness is addressed.

## II. RENORMALIZATION AND RENORMALIZATION GROUP EQUATION OF $\phi^4$ THEORY

The familiar scalar  $\phi^4$  theory is simple, and only one field is involved. Thus it provides an ideal laboratory to study the renormalization group. The unrenormalized Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2} [\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2] - \frac{\lambda_0}{4!} \phi_0^4, \quad (3)$$

we don't discuss the case with spontaneously symmetry breaking, thus the mass square is positive  $m_0^2 > 0$ . The  $\phi^4$  theory in four space-time dimension is renormalizable and the divergences can be absorbed into the redefinition of the fields and coupling parameters. The standard renormalization program is to express the bare quantities in terms of the renormalized ones by

$$\begin{aligned} \phi_0 &= Z_\phi^{1/2} \phi, & \lambda_0 &= Z_\lambda \lambda, \\ m_0^2 &= m^2 - \delta m^2. \end{aligned} \quad (4)$$

The field  $\phi$  and the dimensionless coupling constant  $\lambda$  are multiplicative renormalized, while the renormalization of mass is different: it is additive. We can't write  $m_0^2 = Z_m m^2$  in a conventional way. The reason is that the mass correction is quadratic divergent and others are only logarithmically divergent.

At one-loop order, the self-energy correction is given by

$$-i\Sigma(p^2) = \frac{1}{2}(-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} . \quad (5)$$

The s-channel vertex correction as

$$\Gamma(p^2) = \Gamma(s) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{[(k-p)^2 - m^2 + i\varepsilon]} \frac{i}{(k^2 - m^2 + i\varepsilon)} , \quad (6)$$

The notations can be found in textbook [12]. We omit them to simplify the illustration.

In order to see how the quadratic divergences in the self-energy corrections are produced, let us consider only the integral of the  $\Sigma(p^2)$ ,

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon} = \int \frac{d^4k}{(2\pi)^4} \frac{m^2}{k^2(k^2 - m^2 + i\varepsilon)} + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} . \quad (7)$$

In the above equation, the first part is logarithmically divergent and it is proportional to  $m^2$ . The second part is quadratic divergent and independent of  $m^2$ .

We apply the Pauli-Villars regularization to make the integral finite. For the self-energy correction of Eq. (5), the propagator is modified to

$$\frac{1}{[k^2 - m^2 + i\varepsilon]} \rightarrow \frac{\Lambda^4}{[k^2 - m^2 + i\varepsilon] (k^2 - \Lambda^2 + i\varepsilon)^2} . \quad (8)$$

where  $\Lambda \gg m$  is a large mass parameter. The divergent parts at zero external momentum  $p^2 = 0$  are

$$\Sigma(0) = \frac{\lambda}{32\pi^2} \left[ \Lambda^2 - m^2 \ln \frac{\Lambda^2}{m^2} \right] , \quad \Gamma(0) = i \frac{\lambda^2}{32\pi^2} \ln \frac{\Lambda^2}{m^2} . \quad (9)$$

The basic idea of Fujikawa's renormalization scheme [11] can be demonstrated by introducing a subtraction scale  $\mu$  by the simple relations below,

$$\begin{aligned} \Lambda^2 &= (\Lambda^2 - \mu^2) + \mu^2 , \\ \ln \frac{\Lambda^2}{m^2} &= \ln \frac{\Lambda^2}{\mu^2} + \ln \frac{\mu^2}{m^2} . \end{aligned} \quad (10)$$

The above relations are not just equalities. They represent that the low energy physics is separated from the high energy part. The introduction of scale  $\mu$  can be inferred in the dimensional regularization, for instance,

$$\lambda d^4k \implies \lambda \mu^{4-d} d^d k . \quad (11)$$

If the space-time dimension  $d$  is 2 as done in [3], we need a  $\mu^2$  associated with  $\lambda$  in order to make the coupling constant dimensionless. Thus, the scale  $\mu$  acts a similar role as the renormalization scale in the dimensional regularization.

As the minimal subtraction in dimensional regularization, our scheme is also mass-independent. This provides great advantage in deriving the renormalization group equations. Thus, the one-loop results for renormalization constants are

$$\begin{aligned} Z_\phi &= 1 + \mathcal{O}(\lambda^2), & Z_\lambda &= 1 + \frac{3\lambda}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2}, \\ \delta m^2 &= \frac{\lambda}{32\pi^2} \left( \Lambda^2 - \mu^2 - m^2 \ln \frac{\Lambda^2}{\mu^2} \right). \end{aligned} \quad (12)$$

The choice of scale  $\mu$  is arbitrary and this arbitrariness naturally leads to the renormalization group equations. The unrenormalized field  $\phi_0$  and coupling constant  $\lambda_0$  are independent of  $\mu$ , thus

$$\frac{d\phi_0}{d\ln\mu} = 0, \quad \frac{d\lambda_0}{d\ln\mu} = 0, \quad (13)$$

Two functions can be defined by

$$\gamma_\phi(\lambda) = \frac{1}{2} \frac{1}{Z_\phi} \frac{dZ_\phi}{d\ln\mu}, \quad \beta(\lambda) \equiv \frac{d\lambda}{d\ln\mu} = -\lambda \frac{1}{Z_\lambda} \frac{dZ_\lambda}{d\ln\mu}. \quad (14)$$

The renormalization group equation for mass should be different because the mass renormalization is additive. We don't differentiate the renormalized mass  $m^2$  with respect to  $\ln\mu$  but with  $\mu^2$ . The bare mass is independent of  $\mu$ , thus

$$\mu^2 \frac{dm_0^2}{d\mu^2} = 0, \quad \rightarrow \quad \mu^2 \frac{dm^2}{d\mu^2} = \mu^2 \frac{d(\delta m^2)}{d\mu^2}, \quad (15)$$

From Eq. (12),  $\delta m^2$  contains both  $\mu^2$  and  $m^2$  terms which correspond to quadratic and logarithmic divergences, respectively. Thus, we define two renormalization group functions  $\gamma_\mu$  and  $\gamma_m$  by

$$\mu^2 \frac{dm^2}{d\mu^2} = \gamma_\mu(\lambda) \mu^2 - \gamma_m(\lambda) m^2, \quad (16)$$

A minus sign is added in the  $\gamma_m$  term in order to accord with the conventional definition (differs by a factor of 2). In the adopted renormalization scheme, the functions  $\gamma_\mu$  and  $\gamma_m$  are not explicit  $\mu$ -dependent but are functions of  $\lambda(\mu)$ . Each of them can be interpreted as anomalous dimension of mass.  $\gamma_\mu$  represents anomalous dimension induced by the quadratic divergence.

From our calculations, the renormalization group functions are obtained to be

$$\gamma_\mu = -\frac{\lambda}{32\pi^2}, \quad \gamma_m = -\frac{\lambda}{32\pi^2}. \quad (17)$$

To solve the Eq. (16) is difficult, we restrict our discussions at short-distance where  $\mu^2 \gg m^2$  and only the  $\gamma_\mu$  term is retained. But, even that, we still cannot give an analytic solution because the scale dependence of  $\lambda(\mu)$ . Note that the linear dependence of  $\mu^2$  is more important than the logarithmic dependence when  $\mu$  is large, it is reasonable to neglect the variation of  $\lambda$  with  $\mu$ . Under this approximation, we obtain

$$m^2(\mu) = m^2(\mu_0) - \frac{\lambda}{32\pi^2} (\mu^2 - \mu_0^2) , \quad (18)$$

The renormalized mass square  $m(\mu)^2$  is a linear function of  $\mu^2$ . When  $\mu$  increases,  $m^2$  decreases. This decreasing is ascribed to the negative sign of  $\gamma_\mu$ . There exists a possibility that the induced mass square becomes negative when  $\mu$  is large. This case is related to spontaneous symmetry breaking and we left it for a future study.

The renormalization group equation can also be derived from another way through the  $\Lambda$ -dependence. The fact that renormalized mass is independent of  $\Lambda$  leads to

$$\Lambda^2 \frac{dm_0^2}{d\Lambda^2} = \gamma_\mu(\Lambda)\Lambda^2 - \gamma_m(\Lambda)m^2 . \quad (19)$$

The understanding of renormalization group equation in this way is not new and have implied in the textbook of Zee [13]. Because the renormalized and bare masses satisfies the same evolution equation, we deduce an interpretation: the bare mass is nothing but the renormalized mass by taking the scale  $\mu$  to  $\Lambda$  (or  $\infty$ ) in a cut-off regularization. In other words, the bare mass contains virtual particle momentum from 0 to  $\Lambda$  and the renormalized mass contains momentum from 0 to  $\mu$ . Because the experiment has limited resolution, it seems that the renormalized mass is more important. The above interpretation applies for any bare quantities, i.e. the bare quantities are the renormalized ones at the ultraviolet limit.

The renormalization group equation for Green function can be obtained straightforwardly. Denote  $G_n^0(p, \lambda_0, m_0)$  and  $G_n(p, \lambda, m, \mu)$  by the unrenormalized and renormalized truncated (amputated) connected n-point Green function, respectively. Multiplicative renormalization of field  $\phi$  gives  $G_n^0 = Z_\phi^{-n/2} G_n$ . The unrenormalized Green function does not depend on  $\mu$ , thus  $\mu \frac{d}{d\mu} G_n^0 = 0$ . The renormalization group equation for  $G_n(p, \lambda, m, \mu)$  is

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + 2(\gamma_\mu \mu^2 - \gamma_m m^2) \frac{\partial}{\partial m^2} - n\gamma_\phi \right] G_n(p, \lambda, m, \mu) = 0 . \quad (20)$$

The renormalization group functions  $\beta$ ,  $\gamma$ ,  $\gamma_\mu$ ,  $\gamma_m$  have been defined in Eqs. (14) and (16). The solution of the above equation is similar to the conventional one except that the renormalized mass satisfies the new evolution equation.

### III. THE EVOLUTION OF THE HIGGS MASS

Now, we turn to the Higgs mass. According to Eq. (1) and our treatment of quadratic divergence, the one-loop correction gives

$$\delta m_H^2 = \frac{3}{8\pi^2 v^2} [m_H^2 + 2m_W^2 + m_Z^2 - 4m_t^2] (\Lambda^2 - \mu^2) , \quad (21)$$

The renormalization group equation for the renormalized Higgs mass is obtained by differentiate the above equation with respect to  $\mu^2$ , thus

$$\frac{dm_H^2}{d\mu^2} = \gamma_\mu^H , \quad (22)$$

where the mass anomalous dimension  $\gamma_\mu^H$  is given by

$$\gamma_\mu^H = -\frac{3}{8\pi^2 v^2} [m_H^2 + 2m_W^2 + m_Z^2 - 4m_t^2] . \quad (23)$$

In the anomalous dimension  $\gamma_\mu^H$ , the boson field contribution is negative, while the fermion field part is positive. Compared to the pure scalar field theory, the anomalous dimension  $\gamma_\mu^H$  is proportional to masses of different fields. This is a special property of the SM where all masses are produced from the spontaneously symmetry breaking by unsymmetric vacuum and the fields are coupled to the Higgs proportionally to their masses. Note that it is just this property which makes the Higgs mass stable.

Let us introduce a mass parameter  $m_V$  as

$$m_V = \sqrt{4m_t^2 - 2m_W^2 - m_Z^2} . \quad (24)$$

Here the subscript "V" is borrowed from the name of Veltman, and we may call  $m_V$  by "Veltman mass". If we use the experimental masses,  $m_V \simeq 310$  GeV. Note that the masses appeared in the above equations are renormalized masses rather than the experimentally observed masses.

The solution of the Higgs mass is obtained as

$$m_H^2(\mu) = m_V^2(\mu) + [m_H^2(\mu_0) - m_V^2(\mu_0)] \exp \left\{ -\frac{3}{8\pi^2 v^2} (\mu^2 - \mu_0^2) \right\} . \quad (25)$$

where  $\mu_0$  is an initial energy scale. We have neglected  $\mu$  dependence of  $m_W, m_Z, m_t$  since their dependence is logarithmical. The solution of the Higgs mass is an exponential damping function. It falls very fast. When  $\frac{\mu^2 - \mu_0^2}{v^2} = 8\pi^2 \approx 80$ ,  $\exp \left\{ -\frac{3}{8\pi^2 v^2} (\mu^2 - \mu_0^2) \right\} \approx 0.05$ , the Higgs mass  $m_H^2(\mu)$  is very close to  $m_V^2(\mu)$ . Here, a phenomenon analogous to the Sudakov form factor is reappeared. The exponentiation is because the anomalous dimension  $\gamma_\mu^H$  is proportional to masses of different fields, especially the Higgs mass.

In the short-distance limit, i.e.,  $\mu \rightarrow \infty$ , we have

$$m_H^2 \rightarrow m_V^2 = 4m_t^2 - 2m_W^2 - m_Z^2 . \quad (26)$$

Compared to Eq. (2), Veltman's condition is revived not at the electroweak scale but in the short-distance limit. As we discussed in the previous section, the bare mass is the mass in the short-distance limit. Thus, the bare Higgs mass  $m_H^0 = m_V(\mu = \infty)$  is at the order of the electroweak scale within perturbation theory. There is no quadratic divergence. The Higgs mass at low energy does not receive quadratic divergence and the fine-tuning is not necessary. The problem of the Higgs naturalness aroused by one-loop correction is rescued by radiative corrections themselves.

In the above result, we have neglected the logarithmic corrections. Taking into account them will not modify our conclusion because they are negligible compared to the quadratic terms in the short-distance limit.

#### IV. CONCLUSION

In this study, we have explored the Higgs naturalness problem. Our approach is renormalization group method. Because of the quadratic divergence, the renormalization of the scalar field mass is additive, the conventional multiplicative renormalization is not applicable. A new type renormalization group equation is required for the scalar field mass. Using a subtraction scale, it is possible to study evolution of the mass. An anomalous dimension for mass associated with the quadratic divergence is defined. Then the established renormalization group approach is applied to the Higgs mass. We find a surprising and maybe non-trivial result: the Higgs mass at short-distance is not divergent but an exponential damping function of energy scale. In the short-distance limit, the Higgs mass approaches to a finite mass which we call the "Veltman mass". This mass is at the order of the electroweak scale if the perturbation theory of the SM is valid. The Higgs bare mass is finite, and the fine-tuning is not needed. The SM is peculiar because the couplings are proportional to masses. It is this peculiarity which makes the Higgs mass at the electroweak scale.

In conclusion, the Higgs mass is protected by radiative corrections. The SM Higgs is natural.

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*Note added* After we put the manuscript of this work on arXiv (1104.2735), we saw Fujikawa's paper where his renormalization method is given explicitly [14]. From [14] and the references therein, one can see that there had been many positive attempts to treat renormalization of quadratic divergences of the  $\phi^4$  theory and some formulae are very similar to ours. However, it should be noted that nearly all of them, except Fujikawa's talk at Nankai University, have no effects on our research.

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